

Twisted Dedekind Type Sums Associated with Barnes' Type Multiple Frobenius-Euler l -Functions

Mehmet Cenkci, Yilmaz Simsek, Mumun Can and Veli Kurt
Department of Mathematics, Akdeniz University,
07058-Antalya, Turkey
cenkci@akdeniz.edu.tr, ysimsek@akdeniz.edu.tr,
mcan@akdeniz.edu.tr, vkurt@akdeniz.edu.tr

Abstract : The aim of this paper is to construct new Dedekind type sums. We construct generating functions of Barnes' type multiple Frobenius-Euler numbers and polynomials. By applying Mellin transformation to these functions, we define Barnes' type multiple l -functions, which interpolate Frobenius-Euler numbers at negative integers. By using generalizations of the Frobenius-Euler functions, we define generalized Dedekind type sums and prove corresponding reciprocity law. We also give twisted versions of the Frobenius-Euler polynomials and new Dedekind type sums and corresponding reciprocity law. Furthermore, by using p -adic q -Volkenborn integral and twisted (h, q) -Bernoulli functions, we construct p -adic (h, q) -higher order Dedekind type sums. By using relation between Bernoulli and Frobenius-Euler functions, we also define analogues of Hardy-Berndt type sums. We give some new relations related to these sums as well.

Keywords : Barnes' type multiple Frobenius-Euler numbers and polynomials, Barnes' type multiple Frobenius-Euler l -functions, Bernoulli polynomials and functions, Dedekind sums.

MSC 2000 : 11F20, 11B68, 11M41, 11S40.

1 Introduction, definitions and notations

It is well-known that the classical Dedekind sums $s(h, k)$ first arose in the transformation formula of the logarithm of the Dedekind-eta function. If h and k are coprime integers with $k > 0$, Dedekind sums are defined by

$$s(h, k) = \sum_{a=1}^{k-1} \left(\left(\frac{a}{k} \right) \right) \left(\left(\frac{ha}{k} \right) \right),$$

where

$$\left(\left(x \right) \right) = \begin{cases} x - [x]_G - \frac{1}{2}, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer,} \end{cases}$$

$[x]_G$ being the largest integer $\leq x$. The most important property of Dedekind sums is the reciprocity law, which is given by

$$s(h, k) + s(k, h) = \frac{1}{12} \left(\frac{h}{k} + \frac{k}{h} + \frac{1}{hk} \right) - \frac{1}{4}.$$

For detailed information of Dedekind sums see ([1], [4], [5], [6], [11], [13], [15], [18], [19], [23], [31], [32], [34], [35], [36], [38], [40], [44], [47], [51], [52]).

In this paper, we define new Dedekind type sums related to Frobenius-Euler functions as follows:

Definition 1.1 *Let n , h and k be positive integers with $(h, k) = 1$. We define*

$$S_{n,u}(h, k) = \sum_{a=0}^{k-1} u^{-\frac{ha}{k}} \frac{a}{k} \overline{H}_n \left(\frac{ha}{k}, u \right),$$

where $\overline{H}_n \left(\frac{ha}{k}, u \right)$ denotes Frobenius-Euler function, which is given by Definition 2.6, and u is an algebraic number $\neq 1$.

The most important properties of these sums is the reciprocity law, which is given by the following theorem.

Theorem 1.2 *Let n, h and k be positive integers with $(h, k) = 1$. Then, we have*

$$\begin{aligned} & \left(\frac{u^k}{1-u^k} k^n S_{n,u^k}(h, k) + \frac{u^h}{1-u^h} h^n S_{n,u^h}(k, h) \right) \\ &= \sum_{j=0}^n \binom{n}{j} \frac{u^k}{1-u^k} H_j(u^k) k^j \frac{u^h}{1-u^h} H_{n-j}(u^h) h^{n-j} \\ & \quad + \frac{1}{hk} \frac{u}{1-u} H_{n+1}(u) + \frac{u}{1-u} H_n(u), \end{aligned}$$

where $H_n(u)$ denotes Frobenius-Euler numbers given by (2.1).

Throughout this paper, χ will denote a Dirichlet character of conductor $f = f_\chi$, and χ_0 will be a principle character with conductor $f_{\chi_0} = 1$. We also define Dedekind type sums attached to χ as follows:

Definition 1.3 *Let n, h, k be positive integers with $(h, k) = 1$. Dedekind type sums $S_{n,u^k}(h, k|\chi)$ are defined by*

$$S_{n,u^k}(h, k|\chi) = h^n \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} \frac{a}{k} \overline{H}_n\left(\frac{a}{k} + \frac{b}{h}, u^{hk}\right).$$

Note that if $\chi = \chi_0$ (that is, $f = 1$), then

$$S_{n,u^k}(h, k|1) = \frac{u^{hk} - 1}{u^{hk}} \frac{u^k}{u^k - 1} S_{n,u^k}(h, k).$$

We also note that the Definition 1.3 is different from Nagasaka et.al's definition [34]. In [34], Dedekind sums with character are defined by using Bernoulli polynomials and Bernoulli function. In our definition, we use Frobenius-Euler function $\overline{H}(x, u)$.

Reciprocity law of $S_{n,u^k}(h, k|\chi)$ is given by the following theorem:

Theorem 1.4 *Let χ be a Dirichlet character of conductor $f = f_\chi$ with $f|hk$. Let n, h and k be positive integers with $(h, k) = 1$. Then, we have*

$$\begin{aligned} & (k^n S_{n,u^k}(h, k|\chi) + h^n S_{n,u^h}(k, h|\chi)) \\ &= \frac{1-u^{hk}}{u^{hk}} \frac{u^f}{1-u^f} \left(\frac{1}{hk} H_{n+1,\chi}(u) + H_{n,\chi}(u) \right) \\ & \quad + \frac{u^{hk}}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} \\ & \quad \times \left({}^1H(u^{hk}) hk + {}^2H(u^{hk}) hk + kb + ha \right)^n. \end{aligned}$$

Proofs of Theorem 1.2 and Theorem 1.4 are given in Section 2 and Section 3, respectively.

In this paper, \mathbb{Z}_p , \mathbb{Q}_p , \mathbb{C}_p and \mathbb{C} will, respectively, denote the ring of p -adic integers, the field of p -adic rational numbers, the p -adic completion of the algebraic closure of \mathbb{Q}_p normalized by $|p|_p = p^{-1}$, and the complex field. Let q be an indeterminate such that if $q \in \mathbb{C}$ then $|q| < 1$ and if $q \in \mathbb{C}_p$ then $|1 - q|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log_p q)$ for $|x|_p \leq 1$, where \log_p is the Iwasawa p -adic logarithm function ([20, Chap.4], [22], [24], [26], [49]). We use the notation

$$[x] = [x : q] = \frac{1 - q^x}{1 - q},$$

so that $\lim_{q \rightarrow 1} [x] = x$.

The p -adic q -integral (or q -Volkenborn integral) is originally constructed by Kim [24]. Kim indicated a connection between the q -Volkenborn integral and non-Archimedean combinatorial analysis. The p -adic q -Volkenborn integral is used in mathematical physics, derivation of the functional equation of the q -zeta function and the q -Stirling numbers, and the q -Mahler theory of integration with respect to a ring \mathbb{Z}_p together with Iwasawa's p -adic q - L -function. Recently, many applications of the q -Volkenborn integral have studied by the authors [14], [45], [46], [49], and many mathematicians.

We give some basic properties of p -adic q -Volkenborn integral as follows:

For $g \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{g \mid g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}$, the p -adic q -Volkenborn integral is defined by [22], [24], [26]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N : q]} \sum_{x=0}^{p^N-1} g(x) q^x,$$

where

$$\mu_q(x + p^N \mathbb{Z}_p) = \frac{q^x}{[p^N : q]}$$

is the q -analogue of the Haar measure. For the limiting case $q = 1$,

$$I_1(g) = \lim_{q \rightarrow 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x),$$

with

$$\mu_1(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}$$

is the Haar measure. If $g_1(x) = g(x+1)$, then

$$I_1(g_1) = I_1(g) + g'(0), \quad (1.1)$$

where $g'(0) = \frac{d}{dx}g(x)|_{x=0}$ ([24], [26]).

Let f be any fixed positive integer with $(p, f) = 1$. Then set

$$\begin{aligned} \mathbb{X} &= \mathbb{X}_f = \varprojlim_N (\mathbb{Z}/fp^N\mathbb{Z}), \quad \mathbb{X}_1 = \mathbb{Z}_p, \\ \mathbb{X}^* &= \bigcup_{0 < a < fp} a + fp^n\mathbb{Z}_p, \\ a + fp^n\mathbb{Z}_p &= \{x \in \mathbb{X} : x \equiv a \pmod{fp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ with $0 \leq a < fp^N$. Note that

$$\int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \int_{\mathbb{X}} g(x) d\mu_1(x)$$

for $g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ ([24], [26]).

Let

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \rightarrow \infty} C_{p^n},$$

where $C_{p^n} = \{\zeta : \zeta^{p^n} = 1\}$ is the cyclic group of order p^n . For $\zeta \in T_p$, the function $x \mapsto \zeta^x$ is a locally constant function from \mathbb{Z}_p to \mathbb{C}_p ([21], [28]). By using q -Volkenborn integration, the second author [46] defined generating function of twisted (h, q) -extension of Bernoulli numbers $B_{n,\zeta}^{(h)}(q)$ and polynomials $B_{n,\zeta}^{(h)}(x, q)$ by means of

$$\frac{h \log q + t}{\zeta q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,\zeta}^{(h)}(q) \frac{t^n}{n!}, \text{ and } \frac{h \log q + t}{\zeta q^h e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\zeta}^{(h)}(x, q) \frac{t^n}{n!}, \quad (1.2)$$

respectively. Note that the numbers $B_{n,\zeta}^{(h)}(q)$ are given by [46]

$$B_{0,\zeta}^{(h)}(q) = \frac{h \log q}{\zeta q^h - 1} \text{ and } \zeta q^h \left(B_{\zeta}^{(h)}(q) + 1 \right)^n - B_{n,\zeta}^{(h)}(q) = \delta_{n,1},$$

with the usual convention about replacing $\left(B_{\zeta}^{(h)}(q) \right)^j$ by $B_{j,\zeta}^{(h)}(q)$ in the binomial expansion, where $\delta_{n,1}$ is the Kronecker symbol. If $\zeta \rightarrow 1$, $B_{j,\zeta}^{(h)}(q) \rightarrow B_j^{(h)}(q)$, which are the numbers defined by Kim [27].

In p -adic case, by using p -adic q -Volkenborn integral and twisted (h, q) -Bernoulli functions, we construct p -adic (h, q) -higher order Dedekind type sums as follows:

Definition 1.5 *Let h, a and b be fixed integers with $(a, b) = 1$, and let p be an odd prime such that $p|b$. For $\zeta \in T_p$, we define twisted (h, q) -Dedekind type sums as*

$$s_{m, \zeta}^{(h)}(a, b : q) = \sum_{j=0}^{b-1} \frac{j}{b} \int_{\mathbb{Z}_p} q^{hx} \zeta^x \left(x + \left\{ \frac{ja}{b} \right\} \right)^m d\mu_1(x),$$

where $\{t\}$ denotes the fractional part of a real number t .

Observe that when $h = 1$, $q \rightarrow 1$ and $\zeta \rightarrow 1$, the sum $s_{m, 1}^{(1)}(a, b : 1)$ reduces to p -adic analogue of higher order Dedekind sums $b^m s_{m+1}(a, b)$, defined by Apostol [1]. The main properties of $s_{m, \zeta}^{(h)}(a, b : q)$ will be given in Section 5.

Dedekind sums were generalized by various mathematicians. Here, we list some of them. Apostol [1] defined generalized Dedekind sums $s_n(h, k)$ by

$$s_n(h, k) = \sum_{a=1}^{k-1} \frac{a}{k} \overline{B}_n \left(\frac{ha}{k} \right), \quad (1.3)$$

where n, h, k are positive integers and $\overline{B}_n(x)$ is the n th Bernoulli function, which is defined as follows:

$$\begin{aligned} \overline{B}_n(x) &= B_n(x - [x]_G), \text{ if } n > 1, \\ \overline{B}_1(x) &= \begin{cases} B_1(x - [x]_G), & \text{if } x \notin \mathbb{Z} \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases} \end{aligned} \quad (1.4)$$

where $B_n(x)$ is the Bernoulli polynomial [1], [44], [49]. For odd values of n , these generalized Dedekind sums satisfy a reciprocity law, first proved by Apostol [1]:

$$\begin{aligned} (n+1) & \quad \{hk^n s_n(h, k) + kh^n s_n(k, h)\} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^j B_j h^j B_{n+1-j} k^{n+1-j} + n B_{n+1}, \end{aligned}$$

where $(h, k) = 1$ and B_n is the n th Bernoulli number. Berndt [4] gave a character transformation formula similar to those for the Dedekind η -function and defined Dedekind sums with character $s(h, k; \chi)$ by

$$s(h, k; \chi) = \sum_{a=0}^{kf-1} \chi(a) \overline{B}_{1, \chi} \left(\frac{ha}{k} \right) \overline{B}_1 \left(\frac{a}{kf} \right),$$

for $(h, k) = 1$. Here, χ denotes a primitive character of conductor f and $\overline{B}_{n, \chi}(x)$ is the character Bernoulli function defined as $\overline{B}_{n, \chi}(x) = B_{n, \chi}(x)$ for $0 < x < 1$, where $B_{n, \chi}(x)$ are the character Bernoulli polynomials which are defined as follows ([4], [49]):

$$\sum_{a=0}^{f-1} \frac{\chi(a) t e^{(a+x)t}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n, \chi}(x) \frac{t^n}{n!}.$$

In [19], Gunnells and Sczech defined certain higher-dimensional Dedekind sums that generalize the classical Dedekind sums. By using Barnes' double zeta function Ota [35] defined derivatives of Dedekind sums and proved their reciprocity laws. Using similar method, Nagasaka et.al [34] gave further generalizations of generalized Dedekind sums. Cenkci et.al [15] gave degenerate analogues of classical Dedekind sums and exact generalizations of Berndt's character Dedekind sums to the case of any positive number. By using the p -adic interpolation of certain partial zeta functions, Rosen and Snyder [36] defined p -adic Dedekind sums in the sense of Apostol [1]. They also established the reciprocity law for these new p -adic Dedekind sums via interpolation of corresponding law for generalized Dedekind sums. In [31] and [32], Kudo extended the results of Rosen and Snyder. He defined p -adic continuous function which interpolates higher-order Dedekind sums. In [22], [29], Kim defined q -Bernoulli numbers $\beta_n(q) \in \mathbb{C}$ and q -Bernoulli polynomials $\beta_n(x, q)$ which are different Carlitz's q -Bernoulli numbers [10], [30]. By using these polynomials and an invariant p -adic q -Volkenborn integral on \mathbb{Z}_p , he constructed a p -adic q -analogue of generalized Dedekind sums $b^m s_{m+1}(a, b)$.

In [44], the second author defined new generating functions. By using these functions, he constructed q -Dedekind type sums related to Apostol's Dedekind type sums [1]. By using p -adic q -Volkenborn integral, he [45] constructed p -adic q -higher-order Hardy type sums.

In [27], Kim constructed the new (h, q) -extension of the Bernoulli numbers and polynomials. By applying Mellin transformation to the generating function of the (h, q) -Bernoulli numbers, he defined (h, q) -zeta functions and (h, q) - L -functions, which interpolate (h, q) -Bernoulli numbers at negative integers. By using p -adic q -Volkenborn integral, the distribution property of twisted (h, q) -Bernoulli polynomials is given by the following theorem:

Theorem 1.6 ([46]) *For any positive integer m ,*

$$B_{n,\zeta}^{(h)}(x, q) = m^{n-1} \sum_{a=0}^{m-1} \zeta^a q^{ha} B_{n,\zeta^m}^{(h)}\left(\frac{a+x}{m}, q^m\right) \quad (1.5)$$

for all integers $n \geq 0$.

Observe that for $\zeta \rightarrow 1$, $q \rightarrow 1$ and $h = 1$, we have

$$m^{n-1} \sum_{j=0}^{m-1} B_n\left(x + \frac{j}{m}\right) = B_n(mx). \quad (1.6)$$

The second author [46] gave generating function for twisted (h, q) -extensions of generalized Bernoulli numbers and polynomials associated with a Dirichlet character χ as follows:

$$\begin{aligned} \sum_{a=1}^f \frac{\chi(a) \zeta^a q^{ha} e^{at} (h \log q + t)}{\zeta^f q^{hf} e^{tf} - 1} &= \sum_{n=0}^{\infty} B_{n,\zeta,\chi}^{(h)}(q) \frac{t^n}{n!}, \\ \sum_{a=1}^f \frac{\chi(a) \zeta^a q^{ha} e^{(a+x)t} (h \log q + t)}{\zeta^f q^{hf} e^{tf} - 1} &= \sum_{n=0}^{\infty} B_{n,\zeta,\chi}^{(h)}(x, q) \frac{t^n}{n!}. \end{aligned}$$

Note that

$$\begin{aligned} B_{n,\zeta,\chi}^{(h)}(q) &= f^{n-1} \sum_{j=1}^f \chi(j) \zeta^j q^{hj} B_{n,\zeta}^{(h)}\left(\frac{j}{f}, q^f\right), \\ B_{n,\zeta,\chi}^{(h)}(x, q) &= f^{n-1} \sum_{j=1}^f \chi(j) \zeta^j q^{hj} B_{n,\zeta}^{(h)}\left(\frac{j+x}{f}, q^f\right). \end{aligned} \quad (1.7)$$

Using q -Volkenborn integration, Witt's type formulas for these numbers and polynomials were also given by [46]

$$\begin{aligned} B_{n,\zeta}^{(h)}(q) &= \int_{\mathbb{Z}_p} \zeta^x q^{hx} x^n d\mu_1(x), \\ B_{n,\zeta}^{(h)}(x, q) &= \int_{\mathbb{Z}_p} \zeta^t q^{ht} (x+t)^n d\mu_1(t), \\ B_{n,\zeta,\chi}^{(h)}(q) &= \int_{\mathbb{X}} \chi(x) \zeta^x q^{hx} x^n d\mu_1(x). \end{aligned} \tag{1.8}$$

We note that, if $\zeta \rightarrow 1$ then, $B_{n,\zeta,\chi}^{(h)}(q) \rightarrow B_{n,\chi}^{(h)}(q)$ and $B_{n,\zeta,\chi}^{(h)}(x, q) \rightarrow B_{n,\chi}^{(h)}(x, q)$ which are defined by Kim [27].

Now we summarize our paper as follows:

In Section 2, we construct new generating functions of Frobenius-Euler numbers and polynomials. We give relations between these numbers and polynomials. We also define generating functions of Barnes' type multiple Frobenius-Euler numbers and polynomials. By applying Mellin transformation to these functions, we construct Barnes' type multiple l -functions. We define Dedekind type sums related to the Frobenius-Euler functions. We prove reciprocity laws of these sums. In Section 3, by using Dirichlet character, we give generalizations of the Frobenius-Euler polynomials and numbers. We construct generalized Dedekind type sums and prove corresponding reciprocity law. In Section 4, we give twisted versions of new Dedekind type sums and corresponding reciprocity law. In Section 5, by using p -adic q -Volkenborn integral and twisted (h, q) -Bernoulli functions, we construct p -adic (h, q) -higher order Dedekind type sums. By using relation between Bernoulli and Frobenius-Euler functions, we also define new Hardy-Berndt type sums. We give some new relations related to these sums as well.

2 New Dedekind Type Sums in the Complex Case

Let $F_u(t)$ be the generating function of Frobenius-Euler numbers $H_n(u)$, that is,

$$F_u(t) = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} = \frac{1-u}{e^t - u}, \quad (2.1)$$

([2], [3], [12], [37], [41], [49], [50]). The generating function of Frobenius-Euler polynomials $H_n(x, u)$ can be defined in a natural way by

$$F_u(x, t) = F_u(t) e^{xt} = \sum_{n=0}^{\infty} H_n(x, u) \frac{t^n}{n!} = \frac{1-u}{e^t - u} e^{xt}. \quad (2.2)$$

Now rewriting $F_u(x, t)$, we have

$$\begin{aligned} F_u(x, t) &= \sum_{n=0}^{\infty} H_n(x, u) \frac{t^n}{n!} = \frac{1-u}{e^t - u} e^{xt} = u^{-1} (u-1) e^{xt} \sum_{n=0}^{\infty} (u^{-1} e^t)^n \\ &= \sum_{n=0}^{\infty} u^{-n-1} (u-1) e^{(n+x)t}. \end{aligned}$$

By applying Mellin transform to $F_u(x, t)$,

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_u(x, -t) dt &= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} u^{-n-1} (u-1) \int_0^{\infty} t^{s-1} e^{-(n+x)t} dt \\ &= \frac{u-1}{u} \sum_{n=0}^{\infty} \frac{u^{-n}}{(n+x)^s}, \end{aligned}$$

where $\Gamma(s)$ is the Euler gamma function. The l -function which interpolates Frobenius-Euler numbers at negative integer values, is defined by

$$l(s; u) = \sum_{n=1}^{\infty} \frac{u^{-n}}{n^s} \quad (2.3)$$

for $\operatorname{Re}(s) > 1$ and $u \in \mathbb{C}$ with $|u| \geq 1$. Two-variable l -function is defined by

$$l(s, x; u) = \sum_{n=0}^{\infty} \frac{u^{-n}}{(n+x)^s} \quad (2.4)$$

for $x \neq$ zero or negative integer, $\operatorname{Re}(s) > 1$ and $u \in \mathbb{C}$ with $|u| \geq 1$. So defined two-variable l -function interpolates Frobenius-Euler polynomials $H_n(x, u)$. Indeed, we have

$$\frac{u-1}{u} \sum_{n=0}^{\infty} \frac{u^{-n}}{(n+x)^s} = \frac{u-1}{u} l(s, x; u) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_u(x, -t) dt.$$

For $s = -n$, $n \in \mathbb{Z}$, $n \geq 0$, by using Cauchy residue theorem, we have

$$\frac{u-1}{u} l(-n, x; u) = H_n(x, u). \quad (2.5)$$

In [25], Kim gave the definition of r th Frobenius-Euler polynomials of x with parameters a_1, \dots, a_r as

$$\frac{(1-u)^r}{\prod_{j=1}^r (e^{a_j t} - u)} e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x, u|a_1, \dots, a_r) \frac{t^n}{n!},$$

for complex numbers x, a_1, \dots, a_r, u such that $a_j \neq 0$ for each $j = 1, \dots, r$ and $|u| > 1$. For $x = 0$, the r th Frobenius-Euler polynomials are called as the r th Frobenius-Euler numbers and denoted by $H_n^{(r)}(0, u|a_1, \dots, a_r) = H_n^{(r)}(u|a_1, \dots, a_r)$.

Let x be a complex number, $\operatorname{Re}(x) > 0$ and a_1, \dots, a_r be real numbers such that $a_j \neq 0$ for each $j = 0, \dots, r$. We modify the definition of r th Frobenius-Euler polynomials of x with parameters a_1, \dots, a_r as

$$\prod_{j=1}^r \frac{1 - u^{a_j}}{e^{a_j t} - u^{a_j}} e^{xt} = \sum_{n=0}^{\infty} H_{r,n}(x, u|a_1, \dots, a_r) \frac{t^n}{n!}.$$

Note that for $r = 1$, $H_{1,n}(x, u|a_1) = H_n(x, u^{a_1})$. We have the following identity about $H_{r,n}(x, u|a_1, \dots, a_r)$:

$$H_{r,n}(x, u|a_1, \dots, a_r) = \left({}^1H(u^{a_1})a_1 + \dots + {}^rH(u^{a_r})a_r + x \right)^n,$$

where in the multinomial expansion of $({}^1H(u^{a_1})a_1 + \dots + {}^rH(u^{a_r})a_r + x)^n$ we mean that

$$({}^iH(u))^j = H_j(u) \text{ but } ({}^iH(u))^j ({}^lH(u))^k \neq H_{j+k}(u) \text{ if } i \neq l.$$

This identity can be shown by using the definition of Frobenius-Euler numbers (2.1):

$$\begin{aligned} \prod_{j=1}^r \frac{1 - u^{a_j}}{e^{a_j t} - u^{a_j}} e^{xt} &= \prod_{j=1}^r \left(\sum_{n_j=0}^{\infty} H_{n_j}(u^{a_j}) \frac{(a_j t)^{n_j}}{n_j!} \right) \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \right) \\ &= \sum_{N=0}^{\infty} \frac{({}^1H(u^{a_1})a_1 + \cdots + {}^rH(u^{a_r})a_r + x)^N t^N}{N!}. \end{aligned}$$

Let $F_{r,u}(x, t)$ be the generating function of $H_{r,n}(x, u|a_1, \dots, a_r)$. Then, we have

$$\begin{aligned} F_{r,u}(x, t) &= \sum_{n=0}^{\infty} H_{r,n}(x, u|a_1, \dots, a_r) \frac{t^n}{n!} = \prod_{j=1}^r \frac{1 - u^{a_j}}{e^{a_j t} - u^{a_j}} e^{xt} \\ &= \prod_{j=1}^r (1 - u^{-a_j}) e^{xt} \sum_{n_j=0}^{\infty} u^{-a_j n_j} e^{-a_j n_j t} \\ &= \prod_{j=1}^r (1 - u^{-a_j}) \sum_{n_1, \dots, n_r=0}^{\infty} u^{-(n_1 a_1 + \cdots + n_r a_r)} e^{-(x + n_1 a_1 + \cdots + n_r a_r)t}. \end{aligned}$$

By applying Mellin transformation to $F_{r,u}(x, t)$, we obtain the following integral representation:

$$\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-xt}}{\prod_{j=1}^r (1 - u^{-a_j} e^{-a_j t})} dt = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{u^{-(n_1 a_1 + \cdots + n_r a_r)}}{(x + n_1 a_1 + \cdots + n_r a_r)^s}. \quad (2.6)$$

By (2.6), we give the definition of multiple Frobenius-Euler function $l_r(s, x; u|a_1, \dots, a_r)$ as follows:

Definition 2.1 For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > r$, we define

$$l_r(s, x; u|a_1, \dots, a_r) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{u^{-(n_1 a_1 + \cdots + n_r a_r)}}{(x + n_1 a_1 + \cdots + n_r a_r)^s}$$

for $\operatorname{Re}(x) > 0$, a_1, \dots, a_r positive real numbers and $u \in \mathbb{C}$, $|u| \geq 1$.

Remark 2.2 If we take $r = 1$ and $a_1 = 1$ in above definition, we get Frobenius-Euler l -function (2.4), and in addition if $x = 0$, we get Frobenius-Euler l -function (2.3). If $r = 1$ and $u = 1$, we have Hurwitz zeta function. If $u = 1$, $r = 1$ and $x = 0$, Riemann zeta function is obtained (cf. [22], [25], [26], [39], [41], [42], [43], [49]).

Substituting $s = -n$, $n \in \mathbb{Z}$, $n \geq 0$ in (2.6), by Cauchy residue theorem, we arrive at the following theorem:

Theorem 2.3 For $n \in \mathbb{Z}$, $n \geq 0$, we have

$$\prod_{j=1}^r (1 - u^{-a_j}) l_r(-n, x; u|a_1, \dots, a_r) = H_{r,n}(x, u|a_1, \dots, a_r).$$

We now list some theorems and definitions for the polynomials $H_n(x, u)$, which are needed in the following sections.

Lemma 2.4 For $n \in \mathbb{Z}$, $n \geq 0$, we have

$$H_n(x+1, u) - uH_n(x, u) = (1-u)x^n.$$

By using (2.1) and (2.2), and after some elementary calculations, we have

$$\begin{aligned} \sum_{n=0}^{\infty} m^n \sum_{j=0}^{m-1} \frac{u^{m-j}}{u^m - 1} H_n\left(x + \frac{j}{m}, u^m\right) \frac{t^n}{n!} &= \sum_{j=0}^{m-1} \frac{u^{m-j}}{u^m - 1} \frac{1 - u^m}{e^{mt} - u^m} e^{\left(x + \frac{j}{m}\right)mt} \\ &= \frac{u}{u - 1} \sum_{n=0}^{\infty} H_n(mx, u) \frac{t^n}{n!}. \end{aligned}$$

Therefore, we easily arrive at the following lemma.

Lemma 2.5 For real x and a positive integer m ,

$$m^n \sum_{j=0}^{m-1} u^{m-1-j} H_n\left(x + \frac{j}{m}, u^m\right) = \frac{u^m - 1}{u - 1} H_n(mx, u)$$

for all integers $n \geq 0$.

Definition 2.6 ([12]) Let $H_n(x, u)$ denotes the n th Frobenius-Euler polynomial and let $\overline{H}_n(x, u)$ be defined recursively by

$$\overline{H}_n(x, u) = H_n(x, u), \quad (0 \leq x < 1), \quad \overline{H}_n(x+1, u) = u\overline{H}_n(x, u).$$

With this definition of $\overline{H}_n(x, u)$, it is easily verified that Lemma 2.5 hold for $\overline{H}_n(x, u)$.

Lemma 2.7 For real x and a positive integer m ,

$$m^n \sum_{j=0}^{m-1} u^{m-1-j} \overline{H}_n\left(x + \frac{j}{m}, u^m\right) = \frac{u^m - 1}{u - 1} \overline{H}_n(mx, u)$$

for all integers $n \geq 0$.

Lemma 2.8 For all integers $n \geq 0$ and $(h, k) = 1$,

$$(hk)^n \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \overline{H}_n\left(\frac{a}{k} + \frac{b}{h}, u^{hk}\right) = (u^{hk} - 1) \frac{u}{u - 1} H_n(u).$$

Proof. Using Lemma 2.7, the left hand side of the above equation becomes

$$\begin{aligned} (hk)^n \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(ha+kb)} \overline{H}_n\left(\frac{a}{k} + \frac{b}{h}, u^{hk}\right) \\ = \frac{u^h}{u^h - 1} (u^{hk} - 1) h^n \sum_{b=0}^{h-1} u^{-kb} \overline{H}_n\left(\frac{kb}{h}, u^h\right). \end{aligned}$$

For $b = 0, 1, \dots, h-1$, the residues $kb \bmod h$ are $c = 0, 1, \dots, h-1$. Therefore,

$$\begin{aligned} \frac{u^h}{u^h - 1} (u^{hk} - 1) h^n \sum_{b=0}^{h-1} u^{-kb} \overline{H}_n\left(\frac{kb}{h}, u^h\right) \\ = (u^{hk} - 1) \frac{u}{u^h - 1} h^n \sum_{c=0}^{h-1} u^{h-c-1} \overline{H}_n\left(\frac{c}{h}, u^h\right) = (u^{hk} - 1) \frac{u}{u - 1} H_n(u), \end{aligned}$$

by Lemma 2.7. ■

In the theory of Dedekind sums, the famous relation is reciprocity law, which plays a major role in this theory and other related topics. We now give the proof of main theorem for this section, which is related to reciprocity law for $S_{n,u}(h, k)$. We use similar methods of Ota ([35]) and Nagasaka et.al ([34]) for proving Theorem 1.2.

Proof of Theorem 1.2. For $r = 2$ and $a_1 = k, a_2 = h$,

$$\begin{aligned} l_2(s; u|k, h) &= \sum_{\substack{m, n=0 \\ (m, n) \neq (0, 0)}}^{\infty} \frac{u^{-(km+hn)}}{(km+hn)^s} \\ &= \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \sum_{m', n'=0}^{\infty} '' \frac{u^{-(kb+ha+hk(m'+n'))}}{(kb+ha+hk(m'+n'))^s} \end{aligned}$$

by writing $n = a + kn'$, $m = b + hm'$, where \sum'' means that the summation is taken over all positive integers m', n' except $(m', n') = (0, 0)$ when $a = b = 0$. Then for $M = m' + n'$,

$$\begin{aligned} l_2(s; u|k, h) &= \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \sum_{M=0}^{\infty} ' \frac{(M+1) u^{-(kb+ha+hkM)}}{(kb+ha+hkM)^s} \\ &= \frac{1}{(hk)^s} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{-(kb+ha)} \sum_{M=0}^{\infty} ' \frac{u^{-hkM}}{\left(\frac{kb+ha}{hk} + M\right)^{s-1}} \\ &\quad + \frac{1}{(hk)^s} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{-(kb+ha)} \sum_{M=0}^{\infty} ' \left(1 - \frac{a}{k} - \frac{b}{h}\right) \frac{u^{-hkM}}{\left(\frac{kb+ha}{hk} + M\right)^s}, \quad (2.7) \end{aligned}$$

where \sum' means that the summation is taken over all positive integers M except $M = 0$ when $a = b = 0$. By using (2.4), we obtain

$$\begin{aligned} l_2(s; u|k, h) &= \frac{1}{(hk)^s} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{-(kb+ha)} l\left(s-1, \frac{kb+ha}{hk}; u^{hk}\right) \\ &\quad + \frac{1}{(hk)^s} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{-(kb+ha)} \left(1 - \frac{a}{k} - \frac{b}{h}\right) l\left(s, \frac{kb+ha}{hk}; u^{hk}\right). \end{aligned}$$

By substituting $s = -n$, $n \in \mathbb{Z}$, $n \geq 0$, into (2.7) and using (2.5), we have

$$\begin{aligned} l_2 \quad (-n, u|k, h) &= \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} H_{n+1} \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ &+ \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \left(1 - \frac{a}{k} - \frac{b}{h} \right) H_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right), \end{aligned}$$

where the values a and b in the sums satisfy

$$0 \leq \frac{a}{k} + \frac{b}{h} < 2 \text{ and } \frac{a}{k} + \frac{b}{h} \neq 1.$$

Let B be the set defined by

$$B = \left\{ (a, b) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq a \leq k-1, 0 \leq b \leq h-1, \frac{a}{k} + \frac{b}{h} > 1 \right\}. \quad (2.8)$$

Then by Lemma 2.4 and Definition 2.6, we obtain

$$\begin{aligned} l_2 \quad (-n; u|k, h) &= \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \overline{H}_{n+1} \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ &+ \frac{(hk)^n}{u^{hk} - 1} (1 - u^{hk}) \sum_{(a,b) \in B} u^{hk-(kb+ha)} \left(\frac{a}{k} + \frac{b}{h} - 1 \right)^{n+1} \\ &+ \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \left(1 - \frac{a}{k} - \frac{b}{h} \right) \overline{H}_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ &+ \frac{(hk)^n}{u^{hk} - 1} (1 - u^{hk}) \sum_{(a,b) \in B} u^{hk-(kb+ha)} \left(1 - \frac{a}{k} - \frac{b}{h} \right) \left(\frac{a}{k} + \frac{b}{h} - 1 \right)^n. \end{aligned}$$

Now, by using Lemma 2.7, Lemma 2.8 and Definition 1.1, we get

$$\begin{aligned}
l_2 \quad (-n; u|k, h) &= \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \overline{H}_{n+1} \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\
&+ \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \overline{H}_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\
&- \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \frac{a}{k} \overline{H}_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\
&- \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} u^{hk-(kb+ha)} \frac{b}{h} \overline{H}_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\
&= \frac{1}{hk} \frac{u}{u-1} H_{n+1}(u) + \frac{u}{u-1} H_n(u) \\
&- \frac{u^k}{u^k - 1} k^n S_{n,u^k}(h, k) - \frac{u^h}{u^h - 1} h^n S_{n,u^h}(k, h). \tag{2.9}
\end{aligned}$$

By definition of $l_2(s; u|k, h)$, we have

$$\begin{aligned}
l_2 \quad (-n; u|k, h) &= \frac{H_{2,n}(u|k, h)}{(1 - u^{-k})(1 - u^{-h})} = \frac{({}^1H(u^k)k + {}^2H(u^h)h)^n}{(1 - u^{-k})(1 - u^{-h})} \\
&= \frac{u^k u^h}{(u^k - 1)(u^h - 1)} \sum_{j=0}^n \binom{n}{j} H_j(u^k) H_{n-j}(u^h) k^j h^{n-j}. \tag{2.10}
\end{aligned}$$

By (2.9) and (2.10), we have

$$\begin{aligned}
&\left(\frac{u^k}{1 - u^k} k^n S_{n,u^k}(h, k) + \frac{u^h}{1 - u^h} h^n S_{n,u^h}(k, h) \right) \\
&= \sum_{j=0}^n \binom{n}{j} \frac{u^k}{1 - u^k} H_j(u^k) k^j \frac{u^h}{1 - u^h} H_{n-j}(u^h) h^{n-j} \\
&+ \frac{1}{hk} \frac{u}{1 - u} H_{n+1}(u) + \frac{u}{1 - u} H_n(u).
\end{aligned}$$

Thus, we arrive the desired result. ■

3 Generalized Dedekind Type Sums Attached to a Dirichlet Character

Character generalizations of classical Dedekind sums have been studied by many mathematicians. By using generalized Bernoulli functions attached to character, Berndt [4] defined Dedekind sums with characters for $n = 1$, and proved reciprocity laws by using either Eisenstein series with characters ([4], [5]), integrals such as contour integrals and Riemann-Stieltjes integrals, or the Poisson summation formula ([6]). Nagasaka et.al [34] defined generalized character Dedekind sums which are different from Berndt's definitions for the case $n = 1$, and Cenkci et.al [15] gave exact generalizations of Berndt's sums to the case any positive number. Simsek [38], [40], [44], [47] considered Dedekind sums. He gave several properties of these sums. In [52], Zhang studied the distribution property of a sum analogous to the Dedekind sums by using mean value theorem of the Dirichlet L -function. Xiali and Zhang [51] studied the asymptotic behavior of the Dedekind sums with a weight of Hurwitz zeta function by applying the mean value theorem of the Dirichlet L -function.

To prove Theorem 1.4, we need the following definitions.

Definition 3.1 ([37], [49], [50]) *For a primitive Dirichlet character χ of conductor $f = f_\chi$, generalized Frobenius-Euler numbers attached to χ , $H_{n,\chi}(u)$, $n \in \mathbb{Z}$, $n \geq 0$, are defined by means of*

$$\sum_{n=0}^{\infty} H_{n,\chi}(u) \frac{t^n}{n!} = \sum_{a=0}^{f-1} \frac{\chi(a) (1 - u^f) u^{f-a} e^{at}}{e^{ft} - u^f}. \quad (3.1)$$

With this definition, it is easy to verify that

$$H_{n,\chi}(u) = f^n \sum_{a=0}^{f-1} \chi(a) u^{f-a} H_n\left(\frac{a}{f}, u^f\right).$$

Also, if F is an integer multiple of f , we have

$$\frac{1 - u^F}{u^f - 1} u^f H_{n,\chi}(u) = F^n \sum_{a=0}^{F-1} \chi(a) u^{F-a} H_n\left(\frac{a}{F}, u^F\right). \quad (3.2)$$

Definition 3.2 Let χ be a Dirichlet character of conductor $f = f_\chi$ with $f|hk$. We define the double l -function $l_2(s; u; \chi|k, h)$ with parameters (k, h) , u and χ by

$$l_2(s; u; \chi|k, h) = \sum_{\substack{n, m=0 \\ (n, m) \neq (0, 0)}}^{\infty} \frac{\chi(km + hn) u^{-(km+hn)}}{(km + hn)^s} \quad (3.3)$$

for $s \in \mathbb{C}$, $\text{Re}(s) > 2$, $u \in \mathbb{C}$, $|u| \geq 1$.

We observe that for $u = 1$, (3.3) reduces to double zeta function

$$\tilde{\zeta}_2(s; (k, h), \chi) = \sum_{\substack{n, m=0 \\ (n, m) \neq (0, 0)}}^{\infty} \frac{\chi(km + hn)}{(km + hn)^s},$$

defined in [34]. Also for primitive character $\chi = 1$, (3.3) reduces to the double l -function $l_2(s; u|k, h)$ defined in Section 2.

$l_2(s; u; \chi|k, h)$ can be analytically continued to the whole plane by the following identities:

$$\begin{aligned} l_2(s; u; \chi|k, h) &= \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \sum_{m', n'=0}^{\infty} \frac{\chi(kb + ha) u^{-(kb+ha+hk(m'+n'))}}{(kb + ha + hk(m' + n'))^s} \\ &= \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} l_2(s, kb + ha; u|hk, hk) \end{aligned} \quad (3.4)$$

Now we give proof of Theorem 1.4 as follows:

Proof of Theorem 1.4. By substituting $s = -n$, $n \in \mathbb{Z}$, $n \geq 0$ in (3.4), we have

$$\begin{aligned} l_2(-n; u; \chi|k, h) &= \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} l_2(-n, kb + ha; u|hk, hk) \\ &= \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} \frac{H_{2,n}(kb + ha, u|hk, hk)}{(1 - u^{-hk})^2} \\ &= \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} \\ &\quad \times \frac{({}^1H(u^{hk})hk + {}^2H(u^{hk})hk + kb + ha)^n}{(1 - u^{-hk})^2}. \end{aligned} \quad (3.5)$$

By substituting $m' + n' = M$ in (3.4), we get

$$l_2 \left(s; u; \chi | k, h \right) = \frac{1}{(hk)^s} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} l \left(s-1, \frac{kb+ha}{hk}; u^{hk} \right) \\ + \frac{1}{(hk)^s} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{-(kb+ha)} \left(1 - \frac{a}{k} - \frac{b}{h} \right) l \left(s, \frac{kb+ha}{hk}; u^{hk} \right).$$

Setting $s = -n$, $n \in \mathbb{Z}$, $n \geq 0$ in the above equality yields

$$l_2 \left(-n; u; \chi | k, h \right) \\ = \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{hk-(kb+ha)} H_{n+1} \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ + \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{hk-(kb+ha)} \left(1 - \frac{a}{k} - \frac{b}{h} \right) H_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ = \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{hk-(kb+ha)} \overline{H}_{n+1} \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ + \frac{(hk)^n}{u^{hk} - 1} (1 - u^{hk}) \sum_{(a,b) \in B} \chi(kb + ha) u^{hk-(kb+ha)} \left(\frac{a}{k} + \frac{b}{h} - 1 \right)^{n+1} \\ + \frac{(hk)^n}{u^{hk} - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{hk-(kb+ha)} \left(1 - \frac{a}{k} - \frac{b}{h} \right) \overline{H}_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ + \frac{(hk)^n}{u^{hk} - 1} (1 - u^{hk}) \sum_{(a,b) \in B} \chi(kb + ha) u^{hk-(kb+ha)} \\ \times \left(1 - \frac{a}{k} - \frac{b}{h} \right) \left(\frac{a}{k} + \frac{b}{h} - 1 \right)^n, \quad (3.6)$$

where B is defined by (2.8). From (3.2), we have

$$(hk)^n \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) u^{hk-(kb+ha)} \overline{H}_n \left(\frac{a}{k} + \frac{b}{h}, u^{hk} \right) \\ = \frac{1 - u^{hk}}{u^f - 1} u^f H_{n,\chi}(u), \quad (3.7)$$

since for the values a and b in the sums, we have

$$\left\{ \frac{kb + ha}{hk} - \left[\frac{kb + ha}{hk} \right]_G : 0 \leq a < k, 0 \leq b < h \right\} = \left\{ \frac{i}{hk} : 0 \leq i < hk \right\}.$$

By using (3.7) in (3.6), we obtain

$$\begin{aligned} l_2(-n; u; \chi|k, h) &= \frac{u^f}{1 - u^f} \frac{1}{hk} H_{n+1, \chi}(u) + \frac{u^f}{1 - u^f} H_{n, \chi}(u) \\ &\quad - \frac{u^{hk}}{u^{hk} - 1} (k^n S_{n, u^k}(h, k|\chi) + h^n S_{n, u^h}(k, h|\chi)), \end{aligned}$$

which together with (3.5) completes the proof. ■

4 Twisted Version of Dedekind Type Sums

One of the curious facts about Frobenius-Euler polynomials is the relationship between Bernoulli polynomials. This relationship occurs when $u = \zeta$ is any root of unity. For example, for the generating function of Bernoulli polynomials $B_n(x)$, $n \in \mathbb{Z}$, $n \geq 0$,

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{m-1} \zeta^{-rj} B_n\left(x + \frac{j}{m}\right) = \frac{te^{xt}}{e^t - 1} \sum_{j=0}^{m-1} \zeta^{-rj} e^{jt/m} = \frac{te^{xt}}{\zeta^{-r} e^{t/m} - 1} = \frac{\zeta^r te^{xt}}{e^{t/m} - \zeta^r},$$

where ζ is a primitive m th root of unity and $m \nmid r$. This implies

$$m^{n-1} \sum_{j=0}^{m-1} \zeta^{-rj} B_n\left(x + \frac{j}{m}\right) = \frac{n\zeta^r}{1 - \zeta^r} H_{n-1}(mx, \zeta^r). \quad (4.1)$$

For $m|r$, we have (1.6). Furthermore, multiplying both sides of (4.1) by ζ^{rj} , summing over r and (1.6), we get

$$m^n B_n\left(x + \frac{j}{m}\right) = B_n(mx) + n \sum_{r=1}^{m-1} \zeta^{rj} \frac{H_{n-1}(mx, \zeta^r)}{\zeta^{-r} - 1},$$

where $0 \leq j < m$ ([12]).

We define twisted Dedekind type sums as follows.

Definition 4.1 Let n be a positive integer, h, k be relatively prime positive integers and $\zeta^{hk-1} = 1, \zeta \neq 1$. Then, we define twisted Dedekind sums by

$$S_{n,\zeta}(h, k) = \sum_{a=0}^{k-1} \zeta^{-\frac{ha}{k}} \frac{a}{k} \overline{H}_n \left(\frac{ha}{k}, \zeta \right).$$

Definition 4.2 Let χ be a Dirichlet character of conductor $f = f_\chi$ with $f|hk, n$ be a positive integer and $\zeta^{hk-1} = 1, \zeta \neq 1$. Then we define

$$S_{n,\zeta^k}(h, k|\chi) = h^n \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) \zeta^{-(kb+ha)} \frac{a}{k} \overline{H}_n \left(\frac{a}{k} + \frac{b}{h}, \zeta \right).$$

Observe that for $\zeta = u$ Definition 4.1 and Definition 4.2 reduce to Definition 1.1 and Definition 1.3, respectively.

By substituting $u = \zeta$ in Theorem 1.2 and Theorem 1.4 with $\zeta^{hk-1} = 1, \zeta \neq 1$, we obtain the reciprocity laws for $S_{n,\zeta}(h, k)$ and $S_{n,\zeta^k}(h, k|\chi)$ as follows:

Theorem 4.3 For positive integer n , relatively prime positive integers h, k and $\zeta^{hk-1} = 1, \zeta \neq 1$, we have

$$\begin{aligned} \frac{\zeta^k}{1 - \zeta^k} \quad & k^n S_{n,\zeta^k}(h, k) + \frac{\zeta^h}{1 - \zeta^h} h^n S_{n,\zeta^h}(k, h) \\ &= \sum_{j=0}^n \binom{n}{j} \frac{\zeta^k}{1 - \zeta^k} H_j(\zeta^k) k^j \frac{\zeta^h}{1 - \zeta^h} H_{n-j}(\zeta^h) h^{n-j} \\ &+ \frac{1}{hk} \frac{1}{1 - \zeta} H_{n+1}(\zeta) + \frac{1}{1 - \zeta} H_n(\zeta). \end{aligned}$$

Theorem 4.4 Let n, χ and ζ be as in Definition 4.2. Then

$$\begin{aligned} k^n S_{n,\zeta^k}(h, k|\chi) &+ h^n S_{n,\zeta^h}(k, h|\chi) \\ &= \frac{1 - \zeta}{1 - \zeta^f} \zeta^f \left(\frac{\zeta^{-1}}{hk} H_{n+1,\chi}(u) + \zeta^{-1} H_{n,\chi}(u) \right) \\ &+ \frac{\zeta}{\zeta - 1} \sum_{a=0}^{k-1} \sum_{b=0}^{h-1} \chi(kb + ha) \zeta^{-(kb+ha)} \\ &\times \left({}^1H(\zeta) hk + {}^2H(\zeta) hk + kb + ha \right)^n. \end{aligned}$$

5 (h, q) -Approach to p -adic Twisted Dedekind Type Sums

In this section, we define twisted (h, q) -Dedekind type sums by using twisted (h, q) -Bernoulli polynomials. By using p -adic interpolation of certain partial zeta function, we interpolate these sums to construct twisted (h, q) - p -adic Dedekind sums.

From Definition 1.5 and binomial expansion, we have

$$\begin{aligned} s_{m,\zeta}^{(h)}(a, b : q) &= \sum_{j=0}^{b-1} \frac{j}{b} \int_{\mathbb{Z}_p} q^{hx} \zeta^x \left(x + \left\{ \frac{ja}{b} \right\} \right)^m d\mu_1(x) \\ &= \sum_{j=0}^{b-1} \sum_{c=0}^m \frac{j}{b} \binom{m}{c} \left\{ \frac{ja}{b} \right\}^{m-c} \int_{\mathbb{Z}_p} q^{hx} \zeta^x x^c d\mu_1(x) \end{aligned} \quad (5.1)$$

By using (1.8), we obtain

$$B_{m,\zeta}^{(h)}(x, q) = \sum_{c=0}^m \binom{m}{c} B_{c,\zeta}^{(h)}(q) x^{m-c}. \quad (5.2)$$

By substituting (5.2) into (5.1), we get

$$s_{m,\zeta}^{(h)}(a, b : q) = \sum_{j=0}^{b-1} \sum_{c=0}^m \frac{j}{b} \binom{m}{c} B_{c,\zeta}^{(h)}(q) \left\{ \frac{ja}{b} \right\}^{m-c} = \sum_{j=0}^{b-1} \frac{j}{b} B_{m,\zeta}^{(h)} \left(\left\{ \frac{ja}{b} \right\}, q \right).$$

Throughout this section, ω will denote the Teichmüller character $(\text{mod } p)$ and we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-1/(p-1)}$, and $\zeta \in T_p$.

Theorem 5.1 *Let a, b, p and ζ be as in Definition 1.5. Then there exists a p -adic continuous function $S_{p,\zeta}^{(h)}(s; a, b : q)$ of s on \mathbb{Z}_p which satisfies*

$$S_{p,\zeta}^{(h)}(m; a, b : q) = b^m s_{m,\zeta}^{(h)}(a, b : q)$$

for all positive integers m such that $m + 1 \equiv 0 \pmod{p-1}$.

Proof. Proof of this theorem is similar to that of Theorem 5 of [23] and Theorem 7 of [45]. Let p be an odd prime, j and b positive integers such that $(p, j) = 1$ and $p|b$. Then, we define

$$T_{\zeta}^{(h)}(s; j, b : q) = \omega^{-1}(j) \frac{\langle j \rangle^s}{b} \sum_{k=0}^{\infty} \binom{s}{k} \left(\frac{b}{j} \right)^k B_{k,\zeta}^{(h)}(q), \quad (5.3)$$

for $s \in \mathbb{Z}_p$, where $\langle x \rangle = x\omega^{-1}(x)$. Since

$$\left| \binom{s}{k} \right|_p \leq 1, \quad \left| \frac{b}{j} \right|_p < 1 \text{ and } \left| B_{k,\zeta}^{(h)}(q) \right|_p \leq 1,$$

the sum

$$\sum_{k=0}^{\infty} \binom{s}{k} \left(\frac{b}{j} \right)^k B_{k,\zeta}^{(h)}(q)$$

converges to a continuous function of s in \mathbb{Z}_p .

Substituting $s = m$ in (5.3), we have

$$\begin{aligned} T_{\zeta}^{(h)}(m; j, b : q) &= \omega^{-1}(j) \frac{\langle j \rangle^m}{b} \sum_{k=0}^m \binom{m}{k} \left(\frac{b}{j} \right)^k B_{k,\zeta}^{(h)}(q) \\ &= \omega^{-m-1}(j) b^{m-1} \sum_{k=0}^m \binom{m}{k} \left(\frac{j}{b} \right)^{m-k} B_{k,\zeta}^{(h)}(q) \\ &= \omega^{-m-1}(j) b^{m-1} B_{m,\zeta}^{(h)}\left(\frac{j}{b}, q\right). \end{aligned}$$

If $m+1 \equiv 0 \pmod{p-1}$, then

$$T_{\zeta}^{(h)}(m; j, b : q) = b^{m-1} B_{m,\zeta}^{(h)}\left(\frac{j}{b}, q\right).$$

Consequently, $T_{\zeta}^{(h)}(m; j, b : q)$ is continuous p -adic extension of $b^{m-1} B_{m,\zeta}^{(h)}\left(\frac{j}{b}, q\right)$.

Now, since

$$s_{m,\zeta}^{(h)}(a, b : q) = \sum_{j=0}^{b-1} \frac{j}{b} B_{m,\zeta}^{(h)}\left(\left\{\frac{aj}{b}\right\}, q\right)$$

and

$$T_{\zeta}^{(h)}(m; j, b : q) = b^{m-1} B_{m,\zeta}^{(h)}\left(\frac{j}{b}, q\right),$$

we have

$$b^m s_{m,\zeta}^{(h)}(a, b : q) = \sum_{j=0}^{b-1} j T_{\zeta}^{(h)}(m; (aj)_b, b : q)$$

for $p|b$ and $m+1 \equiv 0 \pmod{p-1}$. ■

In the sequel, we construct twisted (h, q) -character Dedekind type sums. These sums are new and generalize the sums defined by Kudo [31], [32], Rosen and Synder [36] and Kim [23].

Generalization of Definition 1.5 is given by the following definition.

Definition 5.2 *Let a, b be fixed integers with $(a, b) = 1$, and let p be an odd prime such that $p \nmid b$. For a primitive Dirichlet character with conductor $f = f_\chi$ and $\zeta \in T_p$, we define twisted (h, q) -Dedekind sums as*

$$s_{m, \zeta}^{(h)}(a, b : q, \chi) = \sum_{j=0}^{fb-1} \chi(j) \frac{j}{fb} B_{m, \zeta, \chi}^{(h)} \left(\left\{ \frac{aj}{b} \right\}, q \right).$$

We now generalize Theorem 5.1 by character χ . Observe that when $\chi = \chi_0$, the principle character, Definition 5.2 reduces to Definition 1.5, and the following theorem reduces to Theorem 5.1.

Theorem 5.3 *Let a, b, p, χ and ζ be as in Definition 5.2. Then there exists a p -adic continuous function $S_{p, \zeta, \chi}^{(h)}(s; a, b : q)$ of s on \mathbb{X} which satisfies*

$$S_{p, \zeta, \chi}^{(h)}(m; a, b : q) = fb^m s_{m, \zeta}^{(h)}(a, b : q, \chi)$$

for all positive integers m such that $m + 1 \equiv 0 \pmod{p-1}$.

Proof. We follow the similar method in the proof of Theorem 5.1. Let p be an odd prime, j and b positive integers such that $(p, j) = 1$ and $p \nmid b$. For an embedding of the algebraic closure of \mathbb{Q} , $\overline{\mathbb{Q}}$, into \mathbb{C}_p , we may consider the values of a Dirichlet character χ as lying in \mathbb{C}_p . Then we define

$$T_{\zeta, \chi}^{(h)}(s; j, b : q) = \omega^{-1}(j) \frac{\langle j \rangle^s}{b} \sum_{k=0}^{\infty} \binom{s}{k} \left(\frac{b}{j} \right)^k B_{k, \zeta, \chi}^{(h)}(q), \quad (5.4)$$

for $s \in \mathbb{X}$. Since

$$\left| \binom{s}{k} \right|_p \leq 1, \quad \left| \frac{b}{j} \right|_p < 1 \quad \text{and} \quad \left| B_{k, \zeta, \chi}^{(h)}(q) \right|_p \leq 1,$$

$$\sum_{k=0}^{\infty} \binom{s}{k} \left(\frac{b}{j} \right)^k B_{k, \zeta, \chi}^{(h)}(q)$$

converges to a continuous function of s in \mathbb{X} .

Substituting $s = m$ in (5.4), we have

$$\begin{aligned} T_{\zeta, \chi}^{(h)}(m; j, b : q) &= \omega^{-1}(j) \frac{\langle j \rangle^m}{b} \sum_{k=0}^m \binom{m}{k} \left(\frac{b}{j}\right)^k B_{k, \zeta, \chi}^{(h)}(q) \\ &= \omega^{-m-1}(j) b^{m-1} \sum_{k=0}^m \binom{m}{k} \left(\frac{j}{b}\right)^{m-k} B_{k, \zeta, \chi}^{(h)}(q) \\ &= \omega^{-m-1}(j) b^{m-1} B_{m, \zeta, \chi}^{(h)}\left(\frac{j}{b}, q\right). \end{aligned}$$

If $m+1 \equiv 0 \pmod{p-1}$, then

$$T_{\zeta, \chi}^{(h)}(m; j, b : q) = b^{m-1} B_{m, \zeta, \chi}^{(h)}\left(\frac{j}{b}, q\right).$$

Consequently, $T_{\zeta, \chi}^{(h)}(m, j, b : q)$ is continuous p -adic extension of $b^{m-1} B_{m, \zeta, \chi}^{(h)}\left(\frac{j}{b}, q\right)$.

Now, since

$$s_{m, \zeta}^{(h)}(a, b : q, \chi) = \sum_{j=0}^{fb-1} \chi(j) \frac{j}{fb} B_{m, \zeta, \chi}^{(h)}\left(\left\{\frac{aj}{b}\right\}, q\right)$$

and

$$T_{\zeta, \chi}^{(h)}(m; j, b : q) = b^{m-1} B_{m, \zeta, \chi}^{(h)}\left(\frac{j}{b}, q\right),$$

we have

$$fb^m s_{m, \zeta}^{(h)}(a, b : q, \chi) = \sum_{j=0}^{fb-1} j \chi(j) T_{\zeta, \chi}^{(h)}(m; (aj)_b, b : q)$$

for $p|b$ and $m+1 \equiv 0 \pmod{p-1}$. ■

6 Analogues of Hardy-Berndt Type Sums

As mentioned in Section 1, the classical Dedekind sums first arose in the transformation formula of the logarithm of the Dedekind eta function. The logarithms of the classical theta function are studied by Berndt [7] and Goldberg [17] derived the transformation formulas for classical theta-functions.

Arising in the transformation formulas, there are six different arithmetic sums, which are thus similar to Dedekind sums and called as Hardy sums or Berndt's arithmetic sums. For $h, k \in \mathbb{Z}$ with $k > 0$, these six sums are defined as follows:

$$\begin{aligned} S(h, k) &= \sum_{j=1}^{k-1} (-1)^{j+1+\lceil \frac{hj}{k} \rceil_G}, \quad s_1(h, k) = \sum_{j=1}^k (-1)^{\lceil \frac{hj}{k} \rceil_G} \left(\left(\frac{j}{k} \right) \right), \\ s_2(h, k) &= \sum_{j=1}^k (-1)^j \left(\left(\frac{j}{k} \right) \right) \left(\left(\frac{hj}{k} \right) \right), \quad s_3(h, k) = \sum_{j=1}^k (-1)^j \left(\left(\frac{hj}{k} \right) \right), \\ s_4(h, k) &= \sum_{j=1}^{k-1} (-1)^{\lceil \frac{hj}{k} \rceil_G}, \quad s_5(h, k) = \sum_{j=1}^k (-1)^{j+\lceil \frac{hj}{k} \rceil_G} \left(\left(\frac{j}{k} \right) \right). \end{aligned}$$

The analytic and arithmetical properties of these sums were given by Berndt [7], Berndt and Goldberg [8], Can [9], Goldberg [17], Meyer [33], Simsek [38], Sitaramachandrarao [48].

In this section, we show the sums, defined by Definition 1.1, yield new type sums, which we call analogues of Hardy-Berndt type sums.

By taking $m = 2$, $r = 1$ and $\zeta = -1$ in (4.1), we obtain

$$H_n(x, -1) = \frac{2^{n+1}}{n+1} \left(B_{n+1} \left(\frac{x+1}{2} \right) - B_{n+1} \left(\frac{x}{2} \right) \right). \quad (6.1)$$

From Definition 2.6, it is clear that $\overline{H}_n(x+2, -1) = \overline{H}_n(x, -1)$. Since $\overline{B}_n(x)$ is periodic for any integer, (6.1) can be written in terms of these functions as

$$\overline{H}_n(x, -1) = \frac{2^{n+1}}{n+1} \left(\overline{B}_{n+1} \left(\frac{x+1}{2} \right) - \overline{B}_{n+1} \left(\frac{x}{2} \right) \right).$$

From (1.4), it is easy to see that (1.6) is also valid for the functions $\overline{B}_n(x)$, that is, we have

$$m^{n-1} \sum_{j=0}^{m-1} \overline{B}_n \left(x + \frac{j}{m} \right) = \overline{B}_n(mx). \quad (6.2)$$

From (6.2) for $m = 2$, we get

$$\overline{B}_n \left(\frac{x+1}{2} \right) = 2^{1-n} \overline{B}_n(x) - \overline{B}_n \left(\frac{x}{2} \right).$$

We therefore have

$$\overline{H}_n(x, -1) = \frac{2}{n+1} \overline{B}_{n+1}(x) - \frac{2^{n+2}}{n+1} \overline{B}_{n+1}\left(\frac{x}{2}\right). \quad (6.3)$$

Now, taking $u = -1$ in Definition 1.1, we obtain

$$S_{n,-1}(h, k) = \sum_{a=0}^{k-1} (-1)^{\frac{ha}{k}} \frac{a}{k} \overline{H}_n\left(\frac{ha}{k}, -1\right). \quad (6.4)$$

Substituting (6.3) into (6.4) with $x = \frac{ha}{k}$, we have

$$S_{n,-1}(h, k) = \sum_{a=0}^{k-1} (-1)^{\frac{ha}{k}} \frac{a}{k} \left(\frac{2}{n+1} \overline{B}_{n+1}\left(\frac{ha}{k}\right) - \frac{2^{n+2}}{n+1} \overline{B}_{n+1}\left(\frac{ha}{2k}\right) \right). \quad (6.5)$$

By using (6.5), we define the following new sums, which we call analogues of Hardy-Berndt type sums.

Definition 6.1 For $n, h, k \in \mathbb{Z}$ with $(h, k) = 1$ and $n \geq 0$, we define

$$\begin{aligned} HB_{n,0}(h, k) &= \sum_{a=0}^{k-1} (-1)^{\frac{ha}{k}} \frac{a}{k} \overline{B}_{n+1}\left(\frac{ha}{k}\right), \\ HB_{n,1}(h, k) &= \sum_{a=0}^{k-1} (-1)^{\frac{ha}{k}} \frac{a}{k} \overline{B}_{n+1}\left(\frac{ha}{2k}\right). \end{aligned}$$

Remark 6.2 Let k be an odd integer. Then, we have the following relations:

(i) If h is even, then

$$HB_{n,0}(h, k) = s_{n+1}(h, k) \text{ and } HB_{n,1}(h, k) = s_{n+1}(h, 2k),$$

where $s_{n+1}(h, k)$ is given by (1.3).

(ii) Let h be an odd integer. Then

(a) If $n+1$ is even, then

$$HB_{n,1}(h, k) = 2^{-1-n} s_{2,n+1}(h, k) + \frac{n+1}{2^{n+2}} H_n(-1).$$

(b) If $n+1$ is odd, then

$$HB_{n,1}(h, k) = \frac{1}{2} (1 - 2^{-n}) s_{n+1}(h, k) - \frac{1}{4} s_{5,n+1}(h, k) - 2^{-n} s_{3,n+1}(h, k),$$

where $s_{2,n+1}(h, k)$, $s_{3,n+1}(h, k)$ and $s_{5,n+1}(h, k)$ are generalizations of Berndt's arithmetic sums $s_2(h, k)$, $s_3(h, k)$ and $s_5(h, k)$, respectively ([16]).

Conclusion: The conclusion we can most likely draw from above is that the sums given by Definition 1.1 is different from Carlitz, Apostol, Berndt type Dedekind sums, and Definition 1.3 is different from Ota type Dedekind sums. For instance, Carlitz type Dedekind sums are defined by Frobenius-Euler numbers $H_{n-1}(u^{-1})$ as follows [11]:

$$S(h, k : n) = \frac{n}{h^n} \sum_u \frac{H_{n-1}(u^{-1})}{(u-1)(u^{-h}-1)}.$$

In Definition 1.1 and Definition 1.3, we use Frobenius-Euler functions, which provides a different and useful approach to the theory of Dedekind sum.

Definition 1.5 is different from those of Kim, Rosen and Synder and Kudo.

Acknowledgment: This work was supported by Akdeniz University Scientific Research Projects Unit.

References

- [1] T. M. Apostol, Generalized Dedekind sums and transformation formulae of certain Lambert series, *Duke Math. J.* 17 (1950) 147-157.
- [2] T. M. Apostol, On the Lerch zeta function, *Pacific J. Math.* 1 (1951) 161-167.
- [3] T. M. Apostol, Addendum to "On the Lerch zeta function", *Pacific J. Math.* 2 (1952) p.10.
- [4] B. C. Berndt, Character transformation formulae similiar to those for Dedekind eta function, in: *Proc. Sympos. Pure Math.* 24, Amer. Math. Soc., Providence, RI, (1973) 9-30.
- [5] B. C. Berndt, On Eisenstein series with characters and the values of Dirichlet L -functions, *Acta Arith.* 28 (1975) 299-320.
- [6] B. C. Berndt, Reciprocity theorems for Dedekind sums and generalizations, *Adv. Math.* 23 (1977) 285-316.
- [7] B. C. Berndt, Analytic Eisenstein series, theta-functions and series relations in the spirit of Ramanujan, *J. Reine Angew. Math.* 303/304 (1978) 332-365.

- [8] B. C. Berndt, L. A. Goldberg, Analytic properties of arithmetic sums arising in the theory of the classical theta-functions, SIAM J. Math. Anal. (1984) 143-150.
- [9] M. Can, Some arithmetic on the Hardy sums $s_2(h, k)$ and $s_3(h, k)$, Acta Math. Sin. Engl. Ser. 20 (2) 193-200 (2004) 193-200.
- [10] L. Carlitz, q -Bernoulli numbers and polynomials, Duke Math. J. 15 (1948) 987-1000.
- [11] L. Carlitz, The reciprocity theorem for Dedekind sums, Pacific J. Math. 3 (1953) 523-527.
- [12] L. Carlitz, Eulerian numbers and polynomials, Math. Mag. 32 (5) (1959) 247-260.
- [13] L. Carlitz, Generalized Dedekind sums, Math. Zeitschr. 85 (1964) 83-90.
- [14] M. Cenkci, M. Can, V. Kurt, p -adic interpolation functions and Kummer-type congruences for q -twisted and q -generalized twisted Euler numbers, Advan. Stud. Contemp. Math. 9 (2004) 203-216.
- [15] M. Cenkci, M. Can, V. Kurt, Degenerate and character Dedekind sums, submitted.
- [16] M. Cenkci, M. Can, V. Kurt, Generalized Hardy sums, submitted.
- [17] L. A. Goldberg, Transformation of theta-functions and analogues of Dedekind sums, Thesis, University of Illinois, Urbana, 1981.
- [18] E. Grosswald, H. Rademacher, Dedekind Sums, Carus Monograph, no.16 Math. Assoc. Amer., Washington, D. C., 1972.
- [19] P. E. Gunnels, R. Sczech, Evaluation of Dedekind sums, Eisenstein cocycles and special values of L -functions, Duke Math. J. 118 (2003) 229-260.
- [20] K. Iwasawa, Lectures on p -adic L -Functions, Ann. Math. Studies 74 Princeton University Press, Princeton, 1972.
- [21] L. C. Jang, J. H. Kim, T. Kim, D. H. Lee, D. W. Park, On Witt's formula for the Barnes' multiple Bernoulli polynomials, Far East J. Math. Sci 13 (3) (2004) 13-21.

- [22] T. Kim, On a q -analogue of the p -adic log gamma functions and related integrals, J. Number Theory 76 (1999) 320-329.
- [23] T. Kim, A note on p -adic q -Dedekind sums, C. R. Acad. Bulgare Sci. 54 (2001) 37-42.
- [24] T. Kim, q -Volkenborn integration, Russ. J. Math Phys. 19 (2002) 288-299.
- [25] T. Kim, On Euler-Barnes multiple zeta functions, Russ. J. Math. Phys. 10 (2) (2003) 261-267.
- [26] T. Kim, Non-Archimedean q -integrals associated with multiple Changhee q -Bernoulli polynomials, Russ. J. Math. Phys. 10 (2003) 91-98.
- [27] T. Kim, A new approach to q -zeta function, Adv. Stud. Contep. Math. 11 (2) (2005) 157-162.
- [28] T. Kim, L. C. Jang, S.-H. Rim, H. K. Pak, On the twisted q -zeta functions and q -Bernoulli polynomials, Far East J. Appl. Math. 13 (1) (2003) 13-21.
- [29] T. Kim, H. S. Rim, Remark on p -adic q -Bernoulli numbers, Advan. Stud. Contemp Math. 1 (1999) 127-136.
- [30] N. Koblitz, On Carlitz's q -Bernoulli numbers, J. Number Theory 14 (1982) 332-339.
- [31] A. Kudo, On p -adic Dedekind sums (II), Mem. Fac. Sci. Kyushu Univ. Ser. A 45 (2) (1991) 245-284.
- [32] A. Kudo, On p -adic Dedekind sums, Nagoya Math. J. 144 (1996) 155-170.
- [33] J. L. Meyer, Properties of certain integer-valued analogues of Dedekind sums, Acta Arith. LXXXII (3) (1997) 229-242.
- [34] Y. Nagasaka, K. Ota, C. Sekine, Generalizations of Dedekind sums and their reciprocity laws, Acta Arith. 106 (4) (2003) 355-378.

- [35] K. Ota, Derivatives of Dedekind sums and their reciprocity law, J. Number Theory 98 (2003) 280-309.
- [36] K. H. Rosen, W. M. Snyder, p -adic Dedekind sums, J. Reine Angew. Math. 361 (1985) 23-26.
- [37] K. Shiratani, On Euler numbers, Mem. Fac. Sci. Kyushu Univ. Ser. A 27 (1) (1973) 1-5.
- [38] Y. Simsek, Relation between theta-function Hardy sums Eisenstein and Lambert series in the transformation formula of $\log \eta_{g,h}(z)$, J. Number Theory, 99 (2) (2003) 338-360.
- [39] Y. Simsek, On q -analogue of the twisted L -functions and q -twisted Bernoulli numbers, J. Korean Math. Soc. 40 (6) (2003) 963-975.
- [40] Y. Simsek, Generalized Dedekind sums associated with the Abel sum and the Eisenstein and Lambert series. Adv. Stud. Contemp. Math., 9 (2) (2004) 125-137.
- [41] Y. Simsek, On twisted generalized Euler numbers, Bull. Korean Math. Soc., 41 (2) (2004) 299-306.
- [42] Y. Simsek, q -analogue of the twisted l -series and q -twisted Euler numbers, J. Number Theory, 110 (2) (2005) 267-278.
- [43] Y. Simsek, Theorems on twisted L -function and twisted Bernoulli numbers, Advan. Stud. Contemp. Math. 11 (2) (2005) 205-218.
- [44] Y. Simsek, q -Dedekind type sums related to q -zeta function and basic L -series, J. Math. Anal. Appl. 318 (1) (2006) 333-351.
- [45] Y. Simsek, p -adic q -higher-order Hardy-type sums, J. Koren Math. Soc. 43 (1) (2006) 111-131.
- [46] Y. Simsek, Twisted (h, q) -Bernoulli numbers and polynomials related to twisted (h, q) -zeta function and L -function, J. Math. Anal. Appl. in press.
- [47] Y. Simsek, S. Yang, Transformation of four Titchmarsh-type infinite integrals and generalized Dedekind sums associated with Lambert series. Adv. Stud. Contemp. Math., 9 (2) (2004) 195-202.

- [48] R. Sitaramachandrarao, Dedekind and Hardy sums, *Acta Arith.* XLVIII (1978) 325-340.
- [49] H. M. Srivastava, T. Kim and Y. Simsek, q -Bernoulli numbers and polynomials associated with multiple q -zeta functions and basic L -series, *Russ. J. Math. Phys.* 12 (2) (2005) 241–268.
- [50] H. Tsumura, On a p -adic interpolation of the generalized Euler numbers and its applications, *Tokyo J. Math.* 10 (2) (1987) 281-293.
- [51] H. Xiali, W. Zhang, On the mean values of Dedekind sums with the height of Hurwitz zeta function, *J. Math. Anal. Appl.* 240 (1999) 505-517.
- [52] W. Zhang, A sum analogous to the Dedekind sum and its mean value formula, *J. Number Theory* 89 (2001) 1-13.